

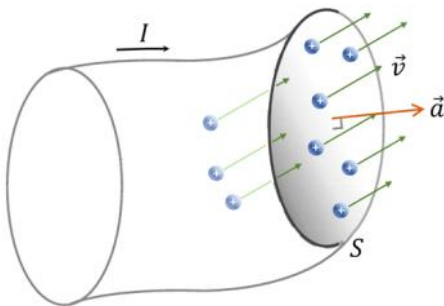
—Chapter 4—

Steady Electric Currents

4-1 Steady Current

A. CURRENT DENSITY

- (1) The amount of charges passing a given point per unit time.



$$I = \frac{\Delta Q}{\Delta t} = \frac{\rho \vec{v} \Delta t \cdot \vec{a}}{\Delta t} = \rho \vec{v} \cdot \vec{a}$$

where ρ is the volume charge density.

Define the current density (current per unit area)

$$\vec{J} = \frac{dI}{da} \hat{n} = \rho \vec{v}$$

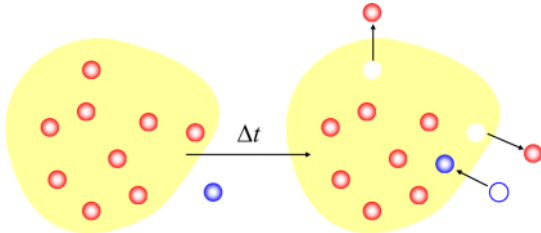
Thus, the current I flowing through any surface S is just the surface integral

$$I = \iint_S \vec{J} \cdot d\vec{a}$$

- (2) If we choose the surface to be closed, Gauss's divergence theorem permits us to express I as an integral over the enclosed volume $\mathcal{V}$:

$$I = \oiint_S \vec{J} \cdot d\vec{a} = \iiint_{\mathcal{V}} \nabla \cdot \vec{J} d\tau$$

Since the charge is leaving the enclosed volume,



$$I = \frac{dQ}{dt} = -\frac{d}{dt} \int_{\mathcal{V}} \rho d\tau$$

so we have

$$\int_V \nabla \cdot \vec{j} d\tau = -\frac{d}{dt} \int_V \rho d\tau = -\int_V \frac{\partial \rho}{\partial t} d\tau$$

This equation gives rise to

$$\nabla \cdot \vec{j} = -\frac{\partial \rho}{\partial t} \Rightarrow \nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0 \dots \dots \text{continuity equation}$$

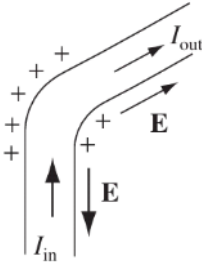
Since charge is never created or destroyed, the charge density ρ and the current density $\vec{j}$ always satisfy the continuity equation.

This equation is also the precise mathematical statement of local charge conservation.

B. STEADY CURRENT

(1) No current source/sink

Suppose that the current into the bend is greater than the current out.



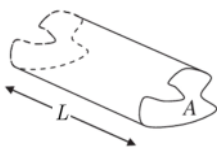
Then charge piles up at the "knee", and this produces a field aiming away from the kink. This field opposes the current flowing in (slowing it down) and promotes the current flowing out (speeding it up) until these currents are equal, at which point there is no further accumulation of charge, and equilibrium is established. Since no charge pilling up occurs, i.e., ρ is constant in time, thus we obtain

$$\frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{j} = 0 \dots \dots \text{no current source/sink}$$

The current is the same all around the circuit, i.e., the current is steady in a wire.

(2) Constant Field

A wire has arbitrary shape of the cross-section A and length L . If we stipulate that the potential is constant over each end, and the potential difference between the ends is ϕ . The field inside the wire is uniform.



COMMENTS:

Suppose that the wire is a cylinder. On the cylindrical surface, the current flows along the z -axis and

$$\vec{j} \cdot \hat{n} = 0$$

and hence we obtain

$$\vec{E} \cdot \hat{n} = 0 \Rightarrow \frac{\partial \varphi}{\partial n} = 0 \text{ on the cylindrical surface}$$

Thus, φ is independent of the variable r and ϕ , i.e., $\varphi = \varphi(z)$. The Laplace's equation reads:

$$\nabla^2 \varphi(z) = \frac{\partial^2 \varphi}{\partial z^2} = 0 \Rightarrow \varphi = Az$$

Using the boundary conditions:

$$\varphi(0) = 0$$

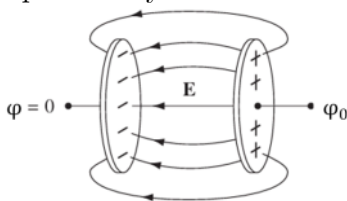
$$\varphi(L) = \varphi_0 = AL \Rightarrow A = \frac{\varphi_0}{L}$$

Thus, the potential and the field are

$$\varphi(z) = \frac{\varphi_0 z}{L}$$

$$\vec{E} = -\nabla \varphi = -\frac{\varphi_0}{L} \hat{z}$$

which is equivalent to a parallel-plate capacitor whose plates are separated by a distance L .



According to Gauss's law,

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0 \nabla \cdot \vec{E}$$

we have

$$\nabla \cdot \vec{j} = -\frac{\partial}{\partial t} (\epsilon_0 \nabla \cdot \vec{E}) = -\epsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} = 0$$

The electric field $\vec{E}$ is constant in time. The current driven by this constant electric field is called the steady current.

4-2 Ohm's Law and Resistance

A. OHM'S LAW

- (1) In most substances, and over a wide range of electric field strengths, we find a *empirical* relation that the current density is proportional to the strength of the electric field that causes it,

$$\vec{J} = \sigma \vec{E} \dots \text{Ohm's law}$$

where σ is called the electric conductivity.

- (2) Drude (1900) model

Assume that electrons are like a classical ideal gas (ignoring the interaction between electrons), moving through a metal subject to an external electric field $\vec{E}$ and experiencing collisions between nuclei.

Thus, the total momentum $\vec{p}$ of a bunch of charge carriers in some volume of a metal is governed by the equation:

$$\frac{d\vec{p}}{dt} = q\vec{E} + \vec{F}_{\text{coll}}$$

During a collision, the impulse ($\vec{J}$) is defined as the integral of the force with respect to time and is equal to the product of the average force and the collision time interval,

$$\vec{J} = \int \vec{F} dt = \vec{F}_{\text{avg}} \Delta t \Rightarrow \vec{F}_{\text{avg}} = \frac{\vec{J}}{\Delta t}$$

According to Newton's second law, we obtain

$$\vec{F}_{\text{avg}} = \frac{\Delta \vec{p}}{\Delta t} \Rightarrow \vec{J} = \Delta \vec{p}$$

If the electron stops dead when a collision happens and the time interval τ is the collision time between nuclei, we then have

$$\begin{aligned} \vec{J} &= \vec{F}_{\text{avg}} \Delta t = \vec{F}_{\text{coll}} \tau \\ \Rightarrow \vec{F}_{\text{coll}} &= \frac{\vec{J}}{\tau} = \frac{\Delta \vec{p}}{\tau} = \frac{0 - m\vec{v}_d}{\tau} = -\frac{m\vec{v}_d}{\tau} \end{aligned}$$

where $\vec{v}_d$ is called the drift velocity and τ is called the relaxation time.

Thus, the equation of motion becomes

$$\frac{d\vec{p}}{dt} = q\vec{E} - \frac{m\vec{v}_d}{\tau}$$

For a steady current, i.e., $d\vec{p}/dt = 0$, we have

$$q\vec{E} = \frac{m\vec{v}_d}{\tau} \Rightarrow \vec{v}_d = \frac{q\tau}{m}\vec{E} = \mu\vec{E}$$

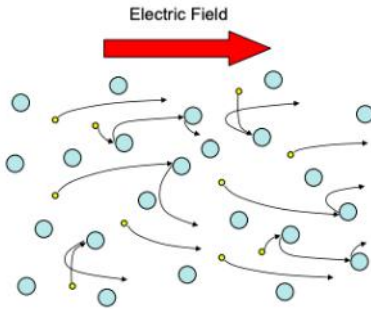
where $\mu = q\tau/m$ is called the mobility.

(3) Physical interpretation of Ohm's law

Using the Drude model, we obtain the current density

$$\begin{aligned}\vec{j} &= \rho\vec{v}_d = \frac{\rho q\tau}{m}\vec{E} = \sigma\vec{E} \\ \Rightarrow \sigma &= \frac{\rho q\tau}{m} = \rho\mu \dots \text{electric conductivity}\end{aligned}$$

Ohm's law describes the motion of charged particles that are accelerated by an electric field but suffer momentum degrading collisions (scattering events) with other particles in a metal.



EXAMPLES:

1. The electron density in copper is $n = 8.5 \times 10^{28} \text{m}^{-3}$. The relaxation time is $\tau \sim 2.5 \times 10^{-14} \text{s}$. Find the electric conductivity of copper.

ANSWER:

Consider the number density n of electrons,

$$\rho = -ne$$

then we have

$$\begin{aligned}\sigma &= \frac{(-ne)(-e)\tau}{m} \\ &= \frac{ne^2\tau}{m} \\ &= \frac{8.5 \times 10^{28} \text{m}^{-3} \times (1.602 \times 10^{-19} \text{C})^2 \times 2.5 \times 10^{-14} \text{s}}{9.109 \times 10^{-31} \text{kg}} \\ &= 6 \times 10^7 (\Omega \text{m})^{-1}\end{aligned}$$

2. The density of copper is 8.935 g/cm^3 . The electron mobility in copper at room temperature is $\mu = 4.4 \times 10^{-3} \text{ m}^2/\text{Vs}$. Find the number of free electrons per copper atom.

ANSWER:

The expression for the electric conductivity of a metal, according to Drude model is as shown below:

$$\sigma = \rho\mu = ne\mu$$

$$\Rightarrow n = \frac{\sigma}{e\mu}$$

$$= \frac{6 \times 10^7 (\Omega\text{m})^{-1}}{1.602 \times 10^{-19} \text{C} \times 4.4 \times 10^{-3} \text{ m}^2/\text{Vs}}$$

$$= 8.5 \times 10^{28} \text{ m}^{-3}$$

The number of copper atom in a cubic meter is

$$N_{\text{Cu}} = \frac{8.935 \times \frac{10^6}{10^3} \text{ kg/m}^3 \times 10^3 \text{ g/kg}}{63.55 \text{ (atomic weight)}} \times N_A$$

$$= \frac{8935000}{63.55} \times 6.023 \times 10^{23}$$

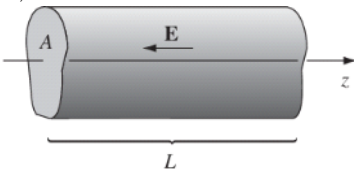
$$= 8.46 \times 10^{28}$$

The number of free electrons in each copper atom is

$$\frac{n}{N_{\text{Cu}}} = \frac{8.5 \times 10^{28}}{8.46 \times 10^{28}} = 1.0$$

B. RESISTANCE

- (1) Consider a piece of homogeneous material of conductivity σ and length L , and uniform cross-section A .



Within the conducting material,

$$\vec{j} = \sigma \vec{E}$$

Since the field inside the wire is uniform, the potential difference $\Delta\phi = V$ between two ends is

$$V = EL$$

The total current is

$$I = \int \vec{j} \cdot d\vec{a} = JA$$

and

$$I = \int \vec{j} \cdot d\vec{a} = \int \sigma \vec{E} \cdot d\vec{a} = \sigma EA = \sigma \frac{V}{L} A$$

Thus, we obtain

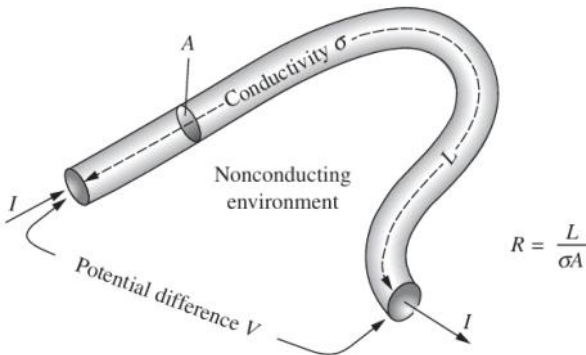
$$J = \frac{I}{A} = \sigma \frac{V}{L} \Rightarrow V = \left(\frac{L}{\sigma A} \right) I \dots \text{Ohm's law}$$

We then define the resistance

$$R = \frac{L}{\sigma A} = \rho_0 \frac{L}{A}$$

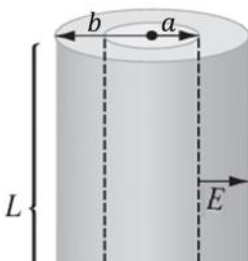
where ρ_0 is called the resistivity.

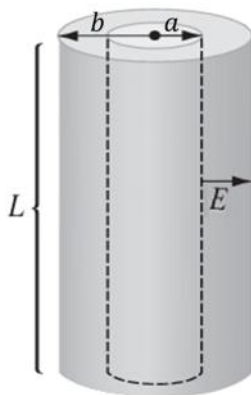
- (2) As long as our conductors are surrounded by a nonconducting medium (air, oil, vacuum, etc.), the resistance R between the terminals doesn't depend on the shape, only on the length of the conductor and its cross-sectional area.



EXAMPLES:

- Two long coaxial metal cylinders (radii a and b) are separated by material of conductivity σ . If they are maintained at a potential difference V , what is the resistance R ?





ANSWER:

The field between the cylinders is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{r}$$

where λ is the charge per unit length on the inner cylinder. The current is therefore

$$I = \int \vec{j} \cdot d\vec{a} = \sigma \int \vec{E} \cdot d\vec{a} = \sigma \frac{\lambda}{2\pi\epsilon_0 r} \cdot 2\pi r L = \frac{\sigma}{\epsilon_0} \lambda L$$

Meanwhile, the potential difference between the cylinders is

$$V = - \int_a^b \vec{E} \cdot d\vec{s} = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{a} \Rightarrow \lambda = \frac{2\pi\epsilon_0}{\ln b/a} V$$

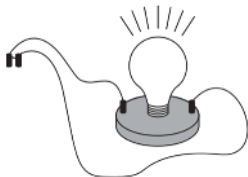
so we have

$$I = \frac{\sigma}{\epsilon_0} \frac{2\pi\epsilon_0}{\ln b/a} V L = \frac{2\pi L \sigma}{\ln b/a} V \Rightarrow R = \frac{\ln b/a}{2\pi L \sigma}$$

4-3 RC Circuit

A. ELECTROMOTIVE FORCE

- (1) Suppose a steady current I , in amperes, flows through a resistor of R .



The flow of current in a resistor involves the dissipation of energy. The power dissipated in a resistor is given by

$$P = I^2 R$$

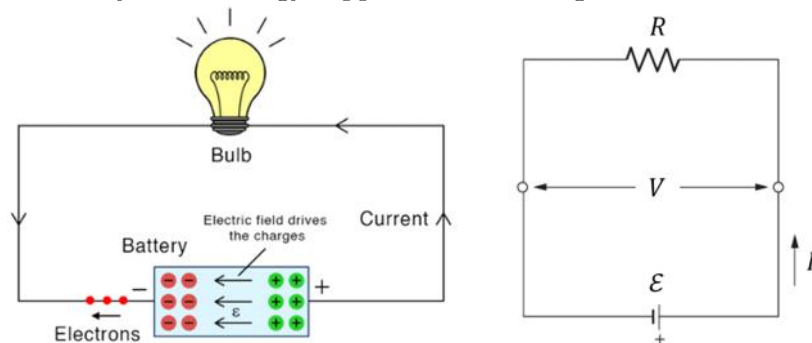
The current will decrease to zero eventually.

- (2) Naturally the steady flow of current in a dc circuit requires some source of energy capable of maintaining the electric field that drives the charge carriers. For historical reasons, we say that these sources of electrical energy impress an electromotive force (EMF) on the system.

OS:

The term electromotive force was coined by Italian physicist and chemist Alessandro Volta, who invented the electric battery in 1800.

A battery is the energy supplied to the charge.



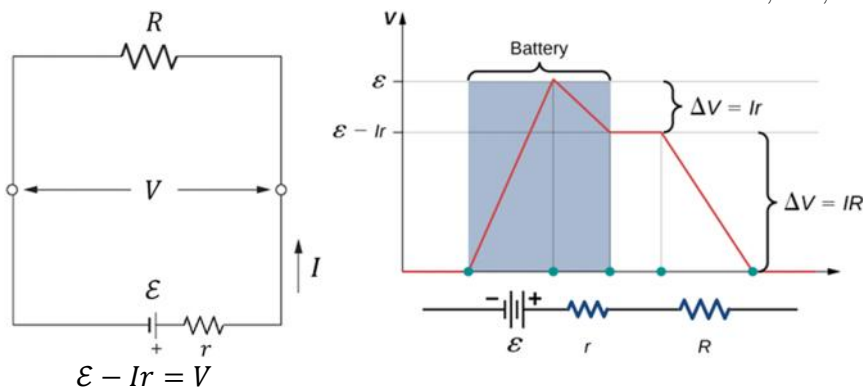
Since

$$\Delta\phi = \oint_C \underbrace{\vec{E}}_{\text{electrostatic field}} \cdot d\vec{s} = \int_S \underbrace{(\nabla \times \vec{E})}_{=0} \cdot d\vec{a} = 0$$

the electric potential difference of the battery is called the electromotive force $\mathcal{E}$,

$$\mathcal{E} = \oint_{\mathcal{C}_{\text{non-electrostatic field}}} \vec{E}' \cdot d\vec{s} \neq 0$$

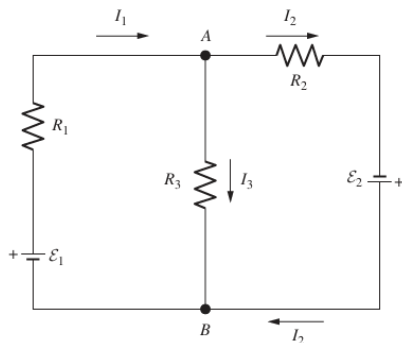
- (3) In practice the electric potential difference will drop only a little below the value $\mathcal{E}$. Thus, r plays the role of an effective "internal resistance" which accounts for loss mechanisms within the source itself, i.e.,



where V is the electric potential difference between the electrodes and is called the voltage V .

EXAMPLES:

1. A circuit contains two batteries without internal resistance with electromotive force $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. What are the currents in this network?



ANSWER:

$$\begin{aligned} I_1 - I_2 - I_3 &= 0 \\ \mathcal{E}_1 - R_1 I_1 - R_3 I_3 &= 0 \end{aligned}$$

$$\mathcal{E}_2 + R_3 I_3 - R_2 I_2 = 0$$

Thus, we obtain

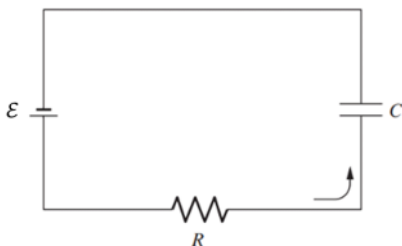
$$I_1 = \frac{\mathcal{E}_1 R_2 + \mathcal{E}_1 R_3 + \mathcal{E}_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

$$I_2 = \frac{\mathcal{E}_2 R_1 + \mathcal{E}_2 R_3 + \mathcal{E}_1 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

$$I_3 = \frac{\mathcal{E}_1 R_2 - \mathcal{E}_2 R_1}{R_1 R_2 + R_2 R_3 + R_1 R_3}$$

B. RC CIRCUIT

- (1) An RC circuit is a circuit containing resistance and capacitance.



The Kirchhoff loop equation for the series RC circuit and the dc current is

$$\mathcal{E} = IR + \frac{q}{C}$$

Since

$$I = \frac{dq}{dt}$$

we have

$$\mathcal{E} = \frac{q}{C} + R \frac{dq}{dt} \Rightarrow \frac{dq}{dt} = \frac{C\mathcal{E}}{RC} - \frac{q}{RC}$$

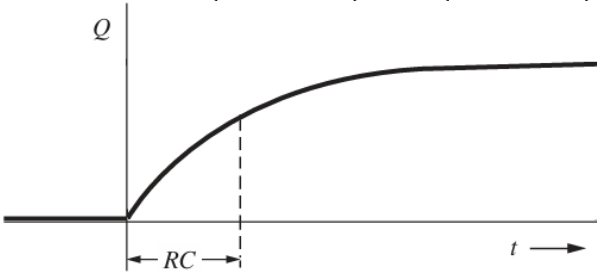
This differential equation can be integrated to find an equation for the charge on the capacitor as a function of time.

$$\begin{aligned} \frac{dq}{C\mathcal{E} - q} &= \frac{dt}{RC} \\ \Rightarrow \int_0^Q \frac{dq}{C\mathcal{E} - q} &= \frac{1}{RC} \int_0^t dt \\ \Rightarrow -\ln(C\mathcal{E} - q) \Big|_0^Q &= \frac{t}{RC} \\ \Rightarrow -\ln \frac{C\mathcal{E} - Q}{C\mathcal{E}} &= \frac{t}{RC} \end{aligned}$$

$$\Rightarrow \frac{C\mathcal{E} - Q}{C\mathcal{E}} = e^{-t/RC}$$

Let $Q_0 = C\mathcal{E}$. We have

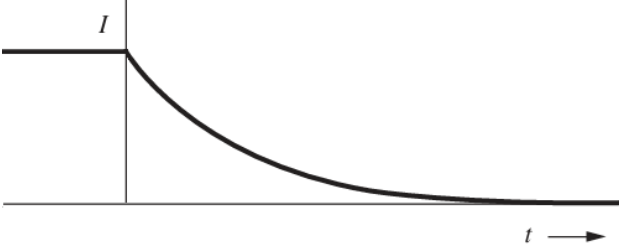
$$Q(t) = C\mathcal{E}(1 - e^{-t/RC}) = Q_0(1 - e^{-t/RC})$$



The quantity RC that appears in the exponent is called the time constant of the circuit.

- (2) The current through the resistor can be found by taking the time derivative of the charge.

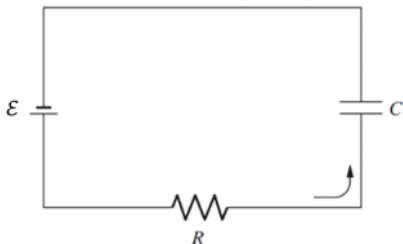
$$I = \frac{dQ}{dt} = \frac{Q_0}{RC} e^{-t/RC} = \frac{\mathcal{E}}{R} e^{-t/RC} = I_0 e^{-t/RC}$$



4-4 Electrostatic Energy

A. ENERGY STORED IN THE ELECTRIC FIELD

- (1) In Electric Circuits, for a simple RC circuit,



the Kirchhoff loop equation for the series RC circuit and the dc current is

$$\mathcal{E} = IR + \frac{Q}{C}$$

and the power supplied by the battery is

$$I\mathcal{E} = I^2R + I\frac{Q}{C}$$

The first term is the power dissipated as heat. The second term is the rate of increase of the electric energy.

Thus, if U is the stored electric energy in the capacitor,

$$\frac{dU}{dt} = I\frac{Q}{C} = \frac{Q}{C}\frac{dQ}{dt} \Rightarrow dU = \frac{Q}{C}dQ$$

Clearly, $U = 0$ when $Q = 0$ (initial condition), then

$$U = \int dU = \frac{1}{C} \int Q dQ = \frac{1}{2C} Q^2$$

Since

$$\frac{Q}{C} = V$$

we obtain

$$U = \frac{1}{2C} C^2 V^2 = \frac{1}{2} CV^2$$

- (2) In Electromagnetism, we would express the electric energy U in terms of $\vec{E}$.

$$\frac{dU}{dt} = \frac{Q}{C} \frac{dQ}{dt} = V \frac{dQ}{dt} = V \frac{d}{dt} \int_V \rho d\tau = \int_V V \frac{\partial \rho}{\partial t} d\tau$$

From the chain rule, we have

$$\frac{d}{dt} \int_{\mathcal{V}} V \rho \, d\tau = \int_{\mathcal{V}} \frac{\partial V}{\partial t} \rho \, d\tau + \int_{\mathcal{V}} V \frac{\partial \rho}{\partial t} \, d\tau = \int_{\mathcal{V}} \frac{\partial V}{\partial t} \rho \, d\tau + V \frac{dQ}{dt}$$

Since

$$V \frac{dQ}{dt} = CV \frac{dV}{dt} = Q \frac{dV}{dt} = \frac{dV}{dt} \int_{\mathcal{V}} \rho \, d\tau = \int_{\mathcal{V}} \frac{\partial V}{\partial t} \rho \, d\tau$$

we obtain

$$\frac{d}{dt} \int_{\mathcal{V}} \rho V \, d\tau = 2 \int_{\mathcal{V}} V \frac{\partial \rho}{\partial t} \, d\tau$$

Thus, we obtain

$$\frac{dU}{dt} = \int_{\mathcal{V}} V \frac{\partial \rho}{\partial t} \, d\tau = \frac{1}{2} \frac{d}{dt} \int_{\mathcal{V}} V \rho \, d\tau \Rightarrow U = \frac{1}{2} \int_{\mathcal{V}} V \rho \, d\tau$$

According to Gauss's law, $\nabla \cdot \vec{E} = \rho/\epsilon_0 \Rightarrow \rho = \epsilon_0 (\nabla \cdot \vec{E})$, we obtain

$$U = \frac{\epsilon_0}{2} \int_{\mathcal{V}} V (\nabla \cdot \vec{E}) \, d\tau$$

Since

$$\nabla \cdot (\vec{E}V) = V (\nabla \cdot \vec{E}) + \underbrace{\vec{E} \cdot \nabla V}_{=E^2} = V (\nabla \cdot \vec{E}) - \underbrace{\vec{E} \cdot \vec{E}}_{=E^2}$$

we obtain

$$U = \frac{\epsilon_0}{2} \left[\int_{\mathcal{V}} \nabla \cdot (\vec{E}V) \, d\tau + \int_{\mathcal{V}} E^2 \, d\tau \right] = \frac{\epsilon_0}{2} \left[\oint_{\mathcal{S}} \vec{E}V \cdot d\vec{a} + \int_{\mathcal{V}} E^2 \, d\tau \right]$$

Since

$$\underbrace{\frac{\vec{E}}{1}}_{\sim \frac{1}{R^2}} \underbrace{\frac{V}{1}}_{\sim \frac{1}{R}} \propto \frac{1}{R^3}$$

and the area of the surface $\mathcal{S}$ grows like R^2 , the surface integral therefore goes to zero as $R \rightarrow \infty$. Thus, we obtain

$$U = \underbrace{\frac{\epsilon_0}{2} \int_{\mathcal{V}} E^2 \, d\tau}_{\text{Electromagnetism}} = \underbrace{\frac{1}{2} CV^2}_{\text{Electric Circuits}}$$

EXAMPLES:

1. Find the energy of a uniformly charged spherical shell of total charge q and radius R .

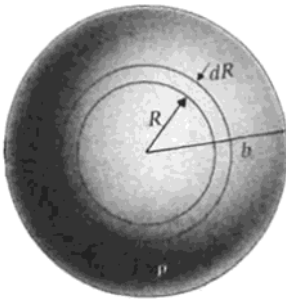
ANSWER:

$$\begin{aligned}
W &= \frac{\epsilon_0}{2} \int E^2 d\tau \\
&= \frac{\epsilon_0}{2} \int \left(\frac{q}{4\pi\epsilon_0 r^2} \right)^2 r^2 \sin\theta dr d\theta d\phi \\
&= \frac{\epsilon_0}{2} \left(\frac{q}{4\pi\epsilon_0} \right)^2 \int_R^\infty \frac{1}{r^2} dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \\
&= \frac{\epsilon_0}{2} \left(\frac{q}{4\pi\epsilon_0} \right)^2 \frac{1}{R} \cdot 2 \cdot 2\pi \\
&= \frac{q^2}{8\pi\epsilon_0 R}
\end{aligned}$$

OS:

$$W = \frac{1}{2} \int \sigma V da = \frac{1}{2} \frac{q}{4\pi\epsilon_0 R} \int \sigma da = \frac{1}{2} \frac{q}{4\pi\epsilon_0 R} q = \frac{q^2}{8\pi\epsilon_0 R}$$

2. Find the energy required to assemble a uniform sphere of charge of radius b and volume charge density ρ .



ANSWER:

- Method I:

From the work required to bring dq in from infinity,

$$\begin{aligned}
V &= \frac{q}{4\pi\epsilon_0 R} \\
dW &= V dq = \frac{q}{4\pi\epsilon_0 R} \rho 4\pi R^2 dR = \frac{q\rho}{\epsilon_0} R dR
\end{aligned}$$

Since

$$q = \frac{4}{3} \pi R^3 \rho$$

we have

$$dW = \frac{\rho}{\epsilon_0} \frac{4}{3} \pi R^3 \rho R dR = \frac{4\pi}{3\epsilon_0} \rho^2 R^4 dR$$

The total work required to assemble a uniform sphere of charge is

$$W = \int dW = \int_0^b \frac{4\pi}{3\epsilon_0} \rho^2 R^4 dR = \frac{4\pi}{15\epsilon_0} \rho^2 b^5$$

- Method II:

From the energy stored in the surrounding electric field,

$$\begin{aligned} \vec{E} &= \begin{cases} \frac{q}{4\pi\epsilon_0 R^2} \hat{r} = \frac{1}{4\pi\epsilon_0 R^2} \frac{4}{3} \pi b^3 \rho \hat{r} = \frac{\rho b^3}{3\epsilon_0 R^2} \hat{r} & , \quad R \geq b \\ \frac{q'}{4\pi\epsilon_0 R^2} \hat{r} = \frac{1}{4\pi\epsilon_0 R^2} \frac{4}{3} \pi R^3 \rho \hat{r} = \frac{\rho R}{3\epsilon_0} \hat{r} & , \quad 0 < R \leq b \end{cases} \\ W &= \frac{\epsilon_0}{2} \int_0^b \left(\frac{\rho R}{3\epsilon_0} \right)^2 4\pi R^2 dR + \frac{\epsilon_0}{2} \int_b^\infty \left(\frac{\rho b^3}{3\epsilon_0 R^2} \right)^2 4\pi R^2 dR \\ &= \frac{2\pi\rho^2}{9\epsilon_0} \left[\int_0^b R^4 dR + b^6 \int_b^\infty \frac{1}{R^2} dR \right] \\ &= \frac{2\pi\rho^2}{9\epsilon_0} \left[\frac{b^5}{5} + b^5 \right] \\ &= \frac{4\pi}{15\epsilon_0} \rho^2 b^5 \end{aligned}$$

OS:

$$\begin{aligned} W &= \frac{1}{2} \int \rho V d\tau = \frac{\rho}{2} \int_0^b V 4\pi R^2 dR \\ V &= - \int_\infty^R \vec{E} \cdot d\vec{s} \\ &= - \int_\infty^b \vec{E} \cdot d\vec{s} - \int_b^R \vec{E} \cdot d\vec{s} \\ &= - \int_\infty^b \frac{\rho b^3}{3\epsilon_0 R^2} dR - \int_b^R \frac{\rho R}{3\epsilon_0} dr \\ &= \frac{\rho}{3\epsilon_0} \left(\frac{b^3}{b} - \frac{b^3}{\infty} - \frac{R^2}{2} + \frac{b^2}{2} \right) \\ &= \frac{\rho}{6\epsilon_0} (3b^2 - R^2) \end{aligned}$$

Thus, we get

$$\begin{aligned}
W &= \frac{\rho}{2} \int_0^b \frac{\rho}{6\epsilon_0} (3b^2 - R^2) 4\pi R^2 dR \\
&= \frac{\pi\rho^2}{3\epsilon_0} \int_0^b (3b^2 R^2 - R^4) dR \\
&= \frac{\pi\rho^2}{3\epsilon_0} \left(b^5 - \frac{b^5}{5} \right) \\
&= \frac{4\pi}{15\epsilon_0} \rho^2 b^5
\end{aligned}$$

3. Find the energy of a point charge near an infinite grounded conducting plane.

ANSWER:

- Method I:

From the work required to bring q in from infinity,

$$\begin{aligned}
W &= \int_{\infty}^d \vec{F}_{\text{ext}} \cdot d\vec{s}, \text{ where } \vec{F}_{\text{ext}} = -\vec{F} \\
&= \frac{1}{4\pi\epsilon_0} \int_{\infty}^d \frac{q^2}{4z^2} dz \\
&= \frac{1}{4\pi\epsilon_0} \left(-\frac{q^2}{4z} \right)_{\infty}^d \\
&= -\frac{q^2}{16\pi\epsilon_0 d}
\end{aligned}$$

- Method II:

From the energy stored in the surrounding electric field,

$$W = \frac{\epsilon_0}{2} \int_{z>0} E^2 d\tau$$

Since $E^2(x, y, -z) = E^2(x, y, z)$, we get

$$W_{\text{analog}} = 2 \frac{\epsilon_0}{2} \int_{z>0} E^2 d\tau = 2W \Rightarrow W = \frac{1}{2} W_{\text{analog}}$$

From method of images, with the two point charges and no conductor, we have

$$W_{\text{analog}} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{2d} = -\frac{q^2}{8\pi\epsilon_0 d}$$

But for a single charge and conducting plane, the energy is half of this:

$$W = \frac{1}{2}W_{\text{analog}} = \frac{1}{2}\left(-\frac{q^2}{8\pi\epsilon_0 d}\right) = -\frac{q^2}{16\pi\epsilon_0 d}$$